TWENTY NINTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 23 April 2016

Second Paper

Time allowed: **Three hours.**

6. Triangle ABC has sides $a = |BC| > b = |AC|$. The points K and H on the segment BC satisfy $|CH| = (a+b)/3$ and $|CK| = (a-b)/3$. If G is the centroid of triangle ABC, prove that $\angle KGH = 90°$.

7. A rectangular array of positive integers has four rows. The sum of the entries in each column is 20. Within each row, all entries are distinct. What is the maximum possible number of columns?

8. Suppose a, b, c are real numbers such that $abc \neq 0$. Determine x, y, z in terms of a, b, c such that

$$bz + cy = a, \quad cx + az = b, \quad ay + bx = c.$$

Prove also that

$$\frac{1-x^2}{a^2} = \frac{1-y^2}{b^2} = \frac{1-z^2}{c^2}.$$

9. Show that the number

$$\left(\frac{251}{\dfrac{1}{\sqrt[3]{252} - 5\sqrt[3]{2}} - 10\sqrt[3]{63}} + \frac{1}{\dfrac{251}{\sqrt[3]{252} + 5\sqrt[3]{2}} + 10\sqrt[3]{63}} \right)^3$$

is an integer and find its value.

10. Let AE be a diameter of the circumcircle of triangle ABC. Join E to the orthocentre, H, of $\triangle ABC$ and extend EH to meet the circle again at D. Prove that the *nine point circle* of $\triangle ABC$ passes through the midpoint of HD.

    Note. The *nine point circle* of a triangle is a circle that passes through the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments that join the orthocentre to the vertices.